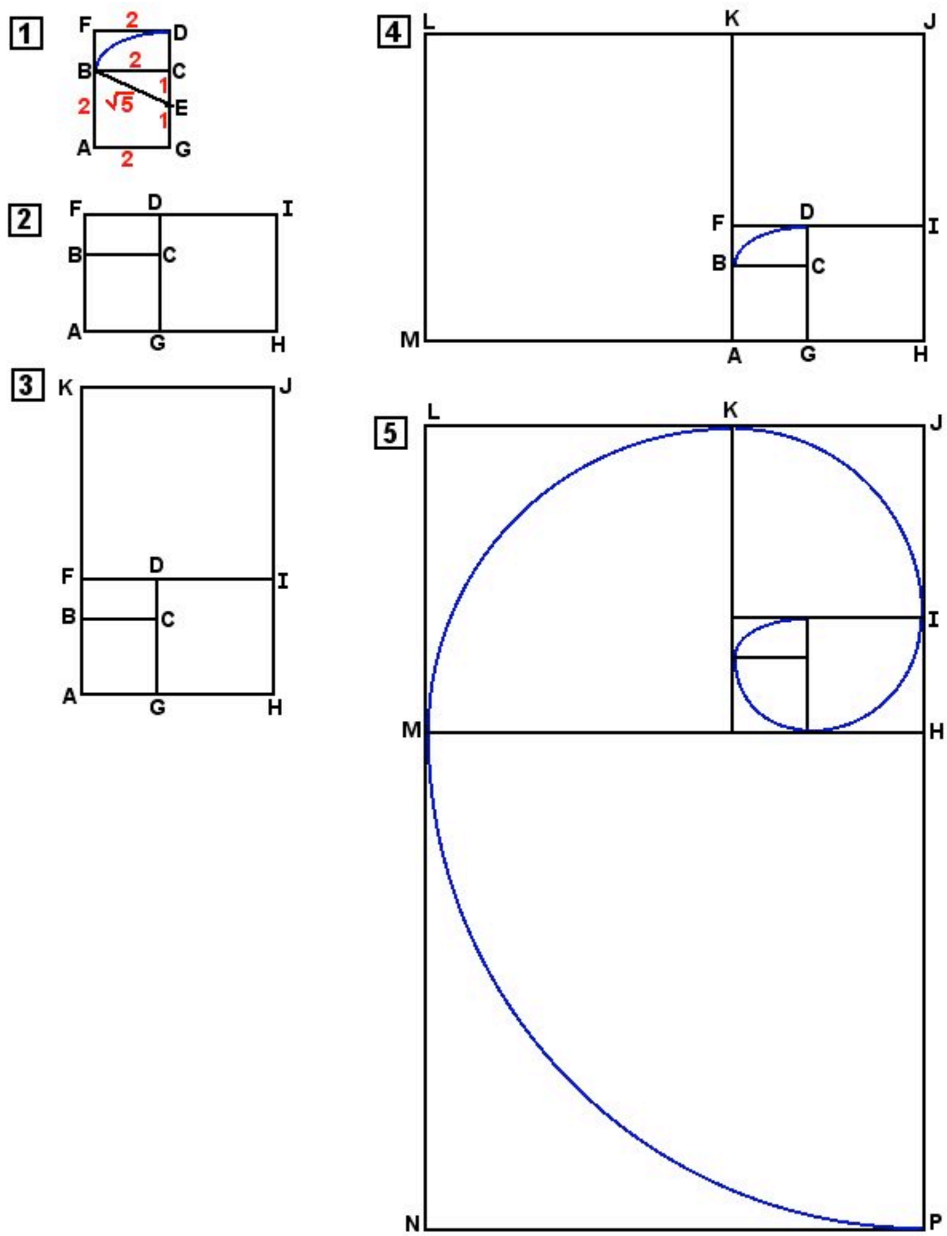
Rationalizing Denominators and the Golden Rectangle and Archimedean Spiral

When I was in school, in the years BC (Before Calculators), I had to learn how to rationalize a denominator. Given the choice between working out $\frac{3+\sqrt{5}}{1+\sqrt{5}}$ or, its equal, $\frac{1+\sqrt{5}}{2}$ without a calculator, but only a table of square roots, the second one was obviously better. However, once calculators came about, the left hand fraction could be worked out fairly well as long as you remembered proper use of brackets. So, at times, when I was teaching how to rationalize a denominator, I thought that maybe this is obsolete knowledge, and should be dropped, just like we no longer teach how to find square roots by hand, without a calculator (this is also something I had to learn in grade nine – I know, I still have my grade 9 exercise books from 48 years ago!).

When I stumbled on the picture on the right, of joined golden ratio rectangles and the Archimedean spiral that results from them, I soon bumped into rationalizing denominators again. In the diagram, number (5) on the lower right is the final answer.



Here is how it is built up, and where the golden ratio comes in. From the last two days, we have seen that any ratio equal to $\frac{1+\sqrt{5}}{2}$, or 1.618033989 is the golden ratio "phi", labeled "Φ" We start, in diagram 1, with a 2 x 2 square ABCG, with E, the mid-point of CG. Thus CE = EG = 1. Now by putting a compass point at E, and radius of EB, draw an arc that meets at GC extended at D. Now extend AB to point F so that AGDF forms a rectangle. Since triangle BCE is a right triangle with sides of 2 and 1, the hypotenuse BE must equal $\sqrt{5}$. Therefore ED is also $\sqrt{5}$ and the length of the rectangle GD is then equal to $1 + \sqrt{5}$. Thus the ratio of length over width of rectangle AGDF is $\frac{1+\sqrt{5}}{2}$ or Φ, the golden ratio.

Next, in diagram 2, I extend FD and AG to form a square GHID. Since the sides are all the same length as DG, they are all equal to $1 + \sqrt{5}$. So the new rectangle formed AHIF has a length, FI, of $2 + 1 + \sqrt{5}$, or $3 + \sqrt{5}$. Its width is FA so that equals $1 + \sqrt{5}$. Hence its length to width ratio is equal to $\frac{3+\sqrt{5}}{1+\sqrt{5}}$. Let's rationalize the denominator to see what this ratio simplifies to. See the next page.

$$\frac{(3+\sqrt{5})}{(1+\sqrt{5})}\times\frac{(1-\sqrt{5})}{(1-\sqrt{5})}=\frac{3-2\sqrt{5}-5}{1-5}=\frac{-2-2\sqrt{5}}{-4}=\frac{-2(1+\sqrt{5})}{-2(2)}=\frac{1+\sqrt{5}}{2}$$

Wow, we get the ratio simplifying to Φ, the golden ratio. Now let's attach another FIJK onto what we have. This is diagram number 3 from the image on page 1. Its sides are all equal to $3+\sqrt{5}$. So the new rectangle AHJK has a length, HJ, of $3+\sqrt{5}+1+\sqrt{5}$ or $4+2\sqrt{5}$. Its width AH is $2+1+\sqrt{5}$ or $3+\sqrt{5}$. Its length ot width ratio is $\frac{4+2\sqrt{5}}{3+\sqrt{5}}$. Is it also a golden rectangle? Let's simplify it by rationalizing its denominator.

$$\frac{(4+2\sqrt{5})}{(3+\sqrt{5})}\times\frac{(3-\sqrt{5})}{(3-\sqrt{5})}=\frac{12+2\sqrt{5}-10}{9-5}=\frac{2+2\sqrt{5}}{4}=\frac{2(1+\sqrt{5})}{2(2)}=\frac{1+\sqrt{5}}{2}$$

Another golden rectangle! Now let's attach another KLMA onto what we have. This is diagram number 4 from the image on page 1. Its sides are all equal to $4+2\sqrt{5}$. Therefore the new rectangle HJLM has a length equal to $4+2\sqrt{5}+3+1\sqrt{5}$, or $7+3\sqrt{5}$. Its width, LM is $4+2\sqrt{5}$, therefore its length to width ratio is $\frac{7+3\sqrt{5}}{4+2\sqrt{5}}$. Is it also a golden rectangle? Let's simplify it by rationalizing its denominator.

$$\frac{(7+3\sqrt{5})}{(4+2\sqrt{5})}\times\frac{(4-2\sqrt{5})}{(4-2\sqrt{5})}=\frac{28-2\sqrt{5}-30}{16-20}=\frac{-2-2\sqrt{5}}{-4}=\frac{-2(1+\sqrt{5})}{-2(2)}=\frac{1+\sqrt{5}}{2}$$

Another golden rectangle!. Finally I added, in diagram (5), the square MNPH, with sides of $7+3\sqrt{5}$ each. This forms a new rectangle JLNP with a length. LN, of $4+2\sqrt{5}+7+3\sqrt{5}$, of $11+5\sqrt{5}$. Its width, LJ is $7+3\sqrt{5}$, so its length to width ratio is $\frac{11+5\sqrt{5}}{7+3\sqrt{5}}$. Is it also a golden rectangle? Let's simplify it by rationalizing its denominator.

$$\frac{(11+5\sqrt{5})}{(7+3\sqrt{5})}\times\frac{(7-3\sqrt{5})}{(7-3\sqrt{5})}=\frac{77+2\sqrt{5}-75}{49-45}=\frac{2+2\sqrt{5}}{4}=\frac{2(1+\sqrt{5})}{2(2)}=\frac{1+\sqrt{5}}{2}$$

Again, a golden rectangle. Finally, another pattern also emerges by looking at the successive sides of the squares added. Observe the sequence below:

2, $1+\sqrt{5}$, $3+\sqrt{5}$, $4+2\sqrt{5}$, $7+3\sqrt{5}$, $11+5\sqrt{5}$. It's a Fibonacci Sequence! Thus, the golden ratio emerges as the ratio between the terms.